数理解析研究所講究録
999 巻 1997 年 39-43

DETERMINANT OF ℓ-ADIC COHOMOLOGY

TAKESHI SAITO (斎藤 毅)

Department of Mathematical Sciences
University of Tokyo (東大 数理)

We consider the following problem.

Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on a smooth scheme U over a field k of characteristic $\neq \ell$. Determine the 1-dimensional ℓ-adic representation

$$\det R\Gamma_c(U_{\bar{k}}, \mathcal{F}) = \bigotimes_q (\wedge^{\dim} H^q_c(U_{\bar{k}}, \mathcal{F}))^{\otimes(-1)^q}$$

of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$.

Under certain mild assumptions, the answers are roughly given as follows.

(1) When the sheaf $\mathcal{F}$ is constant, it is determined by the discriminant of the de Rham cohomology.
(2) In general, it is the tensor product of the following 3 contributions.
(i) That for the constant sheaf raised to its rank-th power.
(ii) The determinant $\det \mathcal{F} = \wedge^{\mathrm{rank}} \mathcal{F}$ "evaluated" at the canonical cycle.
(iii) The Jacobi sum Hecke character determined by the ramification at the boundary.

In this report, only basic ideas will be sketched because the papers [S1], [S2] are already published. There is a Hodge-de Rham versionas announced in [S-T].

1. Basic examples.

A. Fermat curve and Jacobi sum.

We consider the ℓ-adic sheaf $\mathcal{F}$ on $U = \mathbb{P}^1 - \{0, 1, \infty\} = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0, uvw \neq 0\}$ over a field k containing a primitive m-th root of unity defined by the covering by a Fermat curve

$$X = \{(x : y : z) \in \mathbb{P}^2 | x^m + y^m + z^m = 0\} \to \mathbb{P}^1 = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0\}$$
$$(x : y : z) \mapsto (x^m : y^m : z^m)$$

unramified on U. Let $\mathbf{a} = (a, b, c) \in \mathrm{Ker}((\mathbb{Z}/m)^3 \xrightarrow{\text{sum}} \mathbb{Z}/m), a, b, c \neq 0$ be a character of $\mathrm{Gal}(X/\mathbb{P}^1) = \mu_m^3/\text{diag.}$ and let $\mathcal{F}_{\mathbf{a}}$ be the corresponding smooth ℓ-adic sheaf of rank 1 on U. Since $H^q_c(U_{\bar{k}}, \mathcal{F}_{\mathbf{a}}) = 0$ except for $q = 1$ and is of dimension 1 for $q = 1$, the determinant $\det R\Gamma_c(U_{\bar{k}}, \mathcal{F}_{\mathbf{a}})$ is the dual of $H^1_c(U_{\bar{k}}, \mathcal{F}_{\mathbf{a}})$. It is a well-known fact that the 1-dimensional ℓ-adic representation $H^1_c(U_{\bar{k}}, \mathcal{F}_{\mathbf{a}})$ is that defined by the Jacobi sum Hecke character $J_{\mathbf{a}}$. When $k = \mathbb{Q}(\zeta_m)$, for a finite place $p \nmid m$, the algebraic Hecke character $J_{\mathbf{a}}$ is defined by

$$J_{\mathbf{a}}(p) = - \sum_{(u:v:w) \in V(\kappa(v))} \left(\frac{u}{p}\right)_m^a \left(\frac{v}{p}\right)_m^b \left(\frac{w}{p}\right)_m^c$$

where $\left(\frac{}{p}\right)_m$ denotes the m-th power residue symbol at v and $V = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0, uvw \neq 0\}$. The fact above is a consequence of the Grothendieck trace formula. Here we note that a, b, c appearing in the definition of $J_{\mathbf{a}}$ determine the restriction of the character $\mathbf{a}$ to the inertia groups $\mu_m \subset \mu_m^3/\text{diag.}$ at the points $u = 0, v = 0, w = 0$ respectively. As a conclusion, we see the contribution (iii) of the ramification at boundary in this case.

B. Unramified case (cf. [SS]).

In case A above, we only get the contribution of the ramification. However, in a general case, we have contributions (i) and (ii) of global invariants. This is found by Shuji Saito in the case where the base field is finite.

Theorem. (Shuji Saito) *Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on a projective smooth variety X over a finite field k of characteristic $\neq \ell$. Then the action of the geometric Frobenius $Fr_k \in Gal(k_{sep}/k)$ on $\det R\Gamma_c(U_{\bar{k}}, \mathcal{F})$ is given by*

$$\det(Fr_k : R\Gamma_c(X_{\bar{k}}, \mathcal{F})) = \det(Fr_k : R\Gamma_c(X_{\bar{k}}, \mathbb{Q}_\ell))^{\mathrm{rank}\mathcal{F}} \times \det \mathcal{F}(c_X).$$

Here $\det \mathcal{F}(c_X)$ denotes the value of the ℓ-adic character of the arithmetic fundamental group $\pi_1(X)^{ab}$ corresponding to $\det \mathcal{F} = \wedge^{\mathrm{rank}}\mathcal{F}$ evaluated at the image of the canonical class $c_X = (-1)^n c_n(\Omega_X^1) \in CH^n(X), n = \dim X$, by the reciprocity map $CH^n(X) \to \pi_1(X)^{ab}$ of the class field theory.

2. Constant coefficient.

If we assume that our U admits a smooth compactification X such that the complement $D = X - U$ is a divisor with simple normal crossings, then the determinant $\det R\Gamma_c(U_{\bar{k}}, \mathbb{Q}_\ell)$ is the alternating product of $\det R\Gamma(D_{J,\bar{k}}, \mathbb{Q}_\ell)$ where the intersections $D_J = \bigcap_{i \in J} D_i$ of the irreducible components of $D = \bigcup_{i \in I} D_i$ are proper and smooth. In the sequel, we consider the case where $U = X$ is projective and smooth. By Poincaré duality, we see

$$\det R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell)^{\otimes 2} \simeq \mathbb{Q}_\ell(-n\chi)$$

where $n = \dim X$ and χ is the Euler characteristic of $X_{\bar{k}}$. Hence there is a character ϵ of $\mathrm{Gal}(k_{sep}/k)$ of order 2 such that

$$\det R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell) \simeq \epsilon(-\frac{n\chi}{2}).$$

When the dimension n is odd, since the cup-product on H^n is a non-degenerate alternating form, the dimension of H^n and hence the Euler number χ are even and ϵ is trivial. Therefore the only non-trivial problem is to determine ϵ when n is even. The answer is the following.

Theorem 1. *Assume char $k \neq 2$ and let X be a projective smooth variety over k of even dimension $n = 2m$. Then the character ϵ corresponds to the square roots of*

$$(-1)^{m\chi+b^-} \cdot \textit{disc } H^n_{dR}$$

where χ is the Euler number, $b^- = \sum_{q<n} H^q_{dR}(X/k)$ and disc H^n_{dR} is the discriminant of the cup-product of the de Rham cohomology of the middle degree.

Proof is done by taking a Lefschetz pencil and by computing the vanishing cycles by the Picard-Lefschetz formula.

3. With coefficient.

Our result gives an answer under the following rather mild assumption.

(1) The ramification of $\mathcal{F}$ along the boundary is tame. More precisely, we take a smooth compactification X of U such that the complement $D = X - U$ is a divisor with simple normal crossings and, at each irreducible component of D, the pro-p Sylow subgroup of the inertia group acts trivially on the stalk of $\mathcal{F}$.

(2) There is a subring $A \subset k$ finitely generated over $\mathbb{Z}$ such that $\mathcal{F}$ is defined on a model of U on A.

The condition (1) is satisfied if char $k = 0$ and (2) is satisfied if $\mathcal{F}$ is defined geometrically.

Under the hypothesis (2), by the Cebotarev density, the problem is reduced to the residue fields of the maximal ideals of A and hence, for simplicity, we will assume k is finite in the sequel.

First we describe the formula for curve. Let U be a smooth curve over a finite field k of order q and X be the smooth compactification. Let $\mathcal{F}$ be a smooth ℓ-adic sheaf ($\ell \nmid q$) on U at most tamely ramified at the boundary $D = X - U$. For simplicty, we assume that the points $x_i \in D = \{x_i\}$ are rational over k and that, for each x_i, the representation of the inertia group I_i on the stalk of $\mathcal{F}$ is the direct sum $\mathcal{F}|_{I_i} \simeq \bigoplus_j \chi_{i,j}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma(\pi_i^{\frac{1}{q-1}})/\pi_i^{\frac{1}{q-1}}$ where π_i is a uniformizer at x_i. In this case, the product formula [L] of Laumon gives us

Theorem. (Laumon) *Let U be a smooth curve over a finite field k and $\mathcal{F}$ be a smooth ℓ-adic sheaf on U tamely ramified along the boundary satisfying the simplifying assumption above. Then*

$$\det(Fr_k : R\Gamma_c(U_{\bar{k}}, \mathcal{F})) = \det(Fr_k : R\Gamma_c(U_{\bar{k}}, \mathbb{Q}_\ell))^{\mathrm{rank}\mathcal{F}} \times J_{\chi_\mathcal{F}} \times \det \mathcal{F}(c_{X,D}).$$

Here $\chi_\mathcal{F}$ is the family of $M = degD \times rank\mathcal{F}$ characters $(\chi_{i,j})_{x \in D, 1 \leq i \leq \mathrm{rank}\mathcal{F}}$ of $k^\times$ and $J_{\chi_\mathcal{F}}$ denotes the Jacobi sum

$$J_{\chi_\mathcal{F}} = (-1)^M \sum_{(a_{i,j}) \in V(k)} \prod_{i,j} \chi_{i,j}(a_{i,j})$$

where $V = \{(a_{i,j}) \in \mathbb{P}^{M-1} | \sum a_{i,j} = 0, \prod a_{i,j} \neq 0\}$. The relative canonical class $c_{X,D}$ denotes the class

$$-\sum_{x \in U} deg_x\omega \cdot [x] \in \bigoplus_{x \in U} \mathbb{Z}/\{a \in K^\times | a \equiv 1 \bmod D\} = CH^1(X, D)$$

where ω is a rational section of $\Omega^1_X(\log D)$ satisfying $ord_x\omega = 0, res_x = 1$ for $x \in D$ and $\det F(c_{X,D})$ denotes the value of the character of $\pi_1(U)^{\mathrm{ab,tame}}$ corresponding to the rank 1 sheaf $\det \mathcal{F}$ evaluated at the image of $c_{X,D}$ by the reciprocity map $CH^1(X,D) \to \pi_1(U)^{\mathrm{ab,tame}}$ of the class field theory.

Our main result in higher dimension is formally the same as in the case of curve.

Theorem 2. *Let X be a projective smooth variety over a finite field k and U be an open subscheme such that the complement $D = X - U$ is a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on U tamely ramified along the boundary D. Then*

$$\det(Fr_k : R\Gamma_c(U_{\bar{k}}, \mathcal{F})) = \det(Fr_k : R\Gamma_c(U_{\bar{k}}, \mathbb{Q}_\ell))^{\mathrm{rank}\mathcal{F}} \times J_{\chi_{\mathcal{F}}} \times \det \mathcal{F}(c_{X,D})$$

where $J_{\chi_{\mathcal{F}}}$ and $\det \mathcal{F}(c_{X,D})$ are defined below.

The proof is analogous to the constant coefficient case and is done by the induction on dimension by taking a Lefschetz pencil and by computing the vanishing cycle.

In the rest of this report, I explain the idea of the definition of the terms in the right hand side of Theorem 2. The definition of the Jacobi sum is easier. Let X be a smooth compactification of U such that the complement $D = X - U$ is a divisor with simple normal crossings. For simplicity, we assume the constant fields of the components D_i of D are k and the Euler numbers $\chi_i = \sum_q (-1)^q \dim H^q_c(D^*_{i,\bar{k}}, \mathbb{Q}_\ell)$ of $D^*_i = D_i - \cup_{j\neq i} D_j$ are bigger or equal to 0. We also assume for simplicty that for each irreducible component D_i, the representation of the inertia group I_i on the stalk of $\mathcal{F}$ is the direct sum $\mathcal{F}|_{I_i} \simeq \bigoplus_j \chi_{i,j}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma(\pi_i^{\frac{1}{q-1}})/\pi_i^{\frac{1}{q-1}}$ where π_i is a uniformizer of the divisor D_i. Under the above simplifying assumption, we define the Jacobi sum $J_{\chi_{\mathcal{F}}}$ by

$$J_{\chi_{\mathcal{F}}} = (-1)^M \sum_{(a_{i,j,k}) \in V(k)} \prod_{i,j,k} \chi_{i,j}(a_{i,j,k})$$

where i runs the indices of the irreducible components of D, $1 \leq j \leq \mathrm{rank}\mathcal{F}$, $1 \leq k \leq \chi_i$, $M = \mathrm{rank}\mathcal{F} \times \sum_i \chi_i$ and $V = \{(a_{i,j,k}) \in \mathbb{P}^{M-1} | \sum a_{i,j,k} = 0, \prod a_{i,j,k} \neq 0\}$.

Finally I explain the idea of the definition of the relative canonical class $c_{X,D}$ in higher dimension. Note that in the case of curve, the residue $\mathrm{res}_x : \Omega^1_X(\log D) \otimes \kappa(x) \to \kappa(x)$ at $x \in D$ defines a trivialization of the invertible sheaf $\Omega^1_X(\log D)$ at x. In general case, for each irreducible component D_i of the complement $D = X - U$, the residue $\mathrm{res}_i : \Omega^1_X(\log D) \otimes \mathcal{O}_{D_i} \to \mathcal{O}_{D_i}$ defines a partial trivialization of the locally free sheaf $\Omega^1_X(\log D)$ of rank n. This family of partial trivializations enables us to define a refined chern class $c_n(\Omega^1_X(\log D), \mathrm{res})$ as follows. Let's briefly recall a definition of the top chern class $c_n(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ of rank n on a smooth variety X. It is the image of 1 by the composition map

$$\mathbb{Z} \simeq H^n_{\{0\}}(V, \mathcal{K}_n) \to H^n(V, \mathcal{K}_n) \simeq H^n(X, \mathcal{K}_n) = CH_n(X)$$

where V denotes the vector bundle associated to $\mathcal{E}$, $\mathcal{K}_n$ is the Zariski sheaf associated to Quillen's K-group and the last equality is a consequence of the Gersten resolution.

To define the refined chern class $c_n(\Omega^1_X(\log D), \mathrm{res})$, let V be the vector bundle associated to $\Omega^1_X(\log D)$ and we consider complexes

$$\mathcal{K}_{n,X,D} = [\mathcal{K}_{n,X} \to \bigoplus_i \mathcal{K}_{n,D_i}], \mathcal{K}_{n,V,\Delta} = [\mathcal{K}_{n,V} \to \bigoplus_i \mathcal{K}_{n,\Delta}]$$

where $\Delta_i \subset V_{D_i}$ is the inverse image of the 1-section by $\mathrm{res}_i : V_{D_i} \to \mathbb{A}^1_{D_i}$. We define the class as the image of 1 by the composition map

$$\mathbb{Z} \simeq H^n_{\{0\}}(V, \mathcal{K}_{n,V,\Delta}) \to H^n(V, \mathcal{K}_{n,V,\Delta}) \simeq H^n(X, \mathcal{K}_{n,X.D}) = CH^n(X, D).$$

Here the first isomorphism is by the fact $\Delta \cap \{0\} = \emptyset$, the second isomorphism is by the homotopy property of K-cohomolgy and the equality is the definition. Thus $c_{X,D} = (-1)^n c_n(\Omega^1_X(\log D), \mathrm{res}) \in CH^n(X, D)$ is defined. By the reciprocity map $CH^n(X, D) \to \pi_1(U)^{\mathrm{ab,tame}}$, the value $\det \mathcal{F}(c_{X,D})$ of the character of $\pi_1(U)^{\mathrm{ab,tame}}$ corresponding to $\det \mathcal{F}$ evaluated at the image of $c_{X,D}$ is defined. This is the idea of the definition.

More detail will be found in [S1], [S2].

References

[L] G.Laumon, *Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil*, Publ.Math.IHES **65** (1987), 131-210.

[SS] S.Saito, *Functional equations of L-functions of varieties over finite fields*, J.Fac.Sci.Univ.of Tokyo, Sect IA, Math **31** (1984), 287-296.

[S1] T.Saito, *ε-factor of a tamely ramifed sheaf on a variety*, Inv.Math **113** (1993), 389-417.

[S2] ———, *Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies*, J.of Alg.Geom **3** (1994), 411-434.

[S-T] T.Saito-T.Terasoma, *A determinant formula for period integrals*, Proc. of Japan Acad.Ser.A **69** (1993), 131-135.

More references will be found in the lists in [S1], [S2].